ON SYSTEMS OF DIFFERENTIAL EQUATIONS FOR HEAT AND MASS TRANSFER IN CAPILLARY POROUS BODIES

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We consider converting a system of differential equations for heat and mass transfer in capillary porous bodies with transfer potentials T, u, and P to a system of differential equations with mass transfer potentials T, θ, and P. We discuss ways of simplifying the transfer equations and the interrelationship of the heat and mass transfer coefficients.

In recent years solutions of the system of differential equations of heat and mass transfer in capillary porous bodies have been widely used in engineering calculations in drying technology, structural thermophysics, and other branches of industry [1-3]. In addition, various means have been employed for selecting fundamental transfer characteristics. For example, in the drying of moist materials, the temperature T, moisture content u, and pressure P are chosen as the transfer potentials in the majority of cases; in other cases, the temperature T, the moisture transfer potential θ, and the pressure P are chosen. Although the system of differential equations that is obtained is the same, the transfer coefficients and thermodynamic characteristics are different since they refer to different transfer potentials. Hence there arise certain questions concerning the interrelationship among these transfer coefficients. We shall dwell on this in more detail.

1. We consider first the simple case of the heat and moisture transfer in the drying of moist materials in which the gradient of the total pressure is zero (P = const, $\nabla P = 0$). If, as an approximation, we assume that the thermodynamic characteristics (specific heat capacity and thermal gradient coefficient) and the transfer coefficients (thermal diffusivity and moisture diffusion coefficients) do not depend on the coordinates of the body, then, when the variables T and u are chosen as the transfer potentials, the system of differential equations for the heat and moisture transfer assume the form [2]

$$\frac{\partial T}{\partial \tau} = a\nabla^2 T + \frac{\varepsilon r}{c}\frac{\partial u}{\partial \tau}, \tag{1}$$

$$\frac{\partial u}{\partial \tau} = a_m\nabla^2 u + a_m\delta\nabla^2 T. \tag{2}$$

In the derivation of the equations (1) and (2) it is assumed that the body moisture content u is equal to the body liquid moisture content u_2 ($u \approx u_2$) and that the specific heat capacity of the moist body is given by

$$c = c_0 + c_1 u. \tag{3}$$

The thermal gradient coefficient δ is referred to the difference of the moisture contents ($\delta \equiv \delta_u$).

Equations (1) and (2) are derived from the energy and mass conservation equations with the use of the Fourier heat conduction law and the law for the nonisothermal diffusion of moisture in the form

$$j_m = -a_m\rho_0\nabla u - a_m\rho_0\delta\nabla T, \tag{4}$$

where j_m is the moisture flux density.

For the drying of layered materials, consisting of miscellaneous adjacent moist bodies, we use the system of differential equations of heat and moisture transfer with transfer potentials T and θ, since a

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jump in moisture content occurs on a boundary separating individual layers.

The moisture transfer potential θ is a function of the moisture content and the temperature of the body; thus,

$$\theta = \varphi(u, T) \equiv \theta(u, T). \tag{5}$$

We then have

$$d\theta = \left(\frac{\partial\varphi}{\partial u}\right)_T du + \left(\frac{\partial\varphi}{\partial T}\right)_u dT = \frac{1}{C_m} du + \theta'_T dT, \tag{6}$$

where $C_m = (\partial u/\partial\theta)_T$ is the specific isothermal mass capacity (moisture capacity), and $\theta'_T = (\partial\theta/\partial T)_u$ is the temperature coefficient of the mass transfer potential.

Using the relation (6), and assuming that the thermodynamic characteristics C_m and θ'_T do not depend on the coordinates and the time, we can write the system of equations (1) and (2) in the form

$$\frac{\partial T}{\partial\tau} = a'\nabla^2 T + \frac{\varepsilon r C_m}{c'}\frac{\partial\theta}{\partial\tau}, \tag{7}$$

$$\frac{\partial\theta}{\partial\tau} = a'_m\nabla^2\theta + a'_m\delta'\nabla^2 T, \tag{8}$$

where the thermodynamic characteristics and the transfer coefficients are, respectively, given by

$$c' = c + \varepsilon r C_m\theta'_T,\quad \delta' = \frac{\delta}{C_m} + \theta'_T\left(\frac{a'}{a_m} - 1\right), \tag{9}$$

$$a' = \lambda/c'\rho_0,\quad a' = ac/c',\quad a'_m = a_m c'/c. \tag{10}$$

The system of equations (7), (8) is identical with the system of equations (1), (2), except that the heat capacities, the diffusion coefficients, and the thermal gradient coefficients have different magnitudes.

The equation of moisture transfer (8) can be obtained from the mass conservation equation

$$\rho_0\frac{\partial u}{\partial\tau} = -\operatorname{div}\mathbf{j}_m \tag{11}$$

with the use of the nonisothermal moisture conductivity law

$$\mathbf{j}_m = -\lambda\nabla\theta - \lambda_m\delta_\theta\nabla T, \tag{12}$$

where δ_θ is the thermal gradient coefficient referred to the difference of the potentials*:

$$\delta_\theta = \frac{\delta}{C_m} - \theta'_T. \tag{13}$$

The relation (13) is obtained in going from equation (4) to the relations (10) with the aid of the expression (6). Substituting equation (12) into equation (11) and noting that λ_m and δ_θ do not depend on the coordinates, we obtain the differential equation (8) in which the expressions for the coefficients a'_m and δ' are the same as those in equations (9) and (10).

In some papers and books the indices are omitted from the coefficients a', c', a'_m, and δ', resulting in the incorrect notion that the coefficients a, c, a_m, and δ are the same in the systems (1), (2) and (7), (8). It is necessary to keep in mind that the transfer potentials are taken on the basis of the description of the heat and moisture transfer. However, for many moist materials in the process of drying the quantity $C_m\theta'_T$ is small (not the quantity θ'_T itself, but the product $C_m\theta'_T$), so that $\varepsilon r C_m\theta'_T/c \ll 1$; we then obtain

$$c' = c,\quad a' = a,\quad a'_m = a_m. \tag{14}$$

In this case the heat capacities, and the coefficients of heat and moisture diffusion, are the same in the two systems (1), (2) and (7), (8).

If the inequality $\theta'_T(a'_m/a_m - 1)/\delta \ll 1$ holds then $\delta = C_m\delta'$; the reverse case corresponds to $\delta_\theta = 0$, i.e., we can neglect the magnitude of the heat and moisture conductivity (the moisture transfer is described merely by the gradient of the mass transfer), and we then have

*The coefficients δ (grad^{-1}) and δ_θ (grad m^{-1}) have different dimensionalities; therefore, in some papers a coefficient δ'_θ ($\delta'_\theta = C_m\delta_\theta$) is introduced. It has the same dimensionality as δ.

$$C_m\delta' \approx \theta_T' \, (\mathrm{Lu}^{-1} - 1). \tag{15}$$

If the moisture transfer potential does not depend on the temperature ($\theta_T' = 0$), the thermal gradient coefficient δ_θ is not equal to zero ($\delta_\theta = \delta/C_m$), since the second term in the expression (12) describes the thermal moisture conductivity. Analogous relations hold for nonisothermal diffusion in binary gas mixtures in which the mass concentration gradient can be expressed in terms of the gradient of the partial pressure and the gradient of the temperature. In addition, it is necessary to take into account the thermal diffusion or Soret effect (for details, see [4], Chap. 8).

By passing to the limit in the system of equations (1), (2), we obtain, as special cases, the Fourier differential equation of heat conduction and Fick's differential equation of diffusion.

2. Under intense evaporation of moisture inside a capillary porous body, a gradient of the total pressure appears, subject to whose action a vapor transfer takes place of filtrational motion type. The mass flux density j_f of such a transfer is given, based on Darcy's Law, by the equation

$$j_f = j_1 + j_3 = -k_p \nabla P. \tag{16}$$

The differential equation describing the pressure field in the body is derived from the balance equation for the mass of moist air in the pores and capillaries of the body

$$\rho_0 \frac{\partial (u_1 + u_3)}{\partial \tau} = -\operatorname{div} j_f - \varepsilon \rho_0 \frac{\partial u}{\partial \tau} \tag{17}$$

with the aid of the relation (16).

If, as an approximation, we assume that the moist air (vapor—gas mixture) in the capillaries and pores of the body obeys Clapeyron's equation of state, and if we neglect swelling of the capillary walls, we can write

$$\rho_0 d (u_1 + u_3) = \frac{M\Pi b}{RT} \left(dP - \frac{P}{T} \, dT + \frac{P}{b} \, db \right), \tag{18}$$

where $b = b_1 + b_3$ (saturation of the pores and capillaries by the vapor b_1 and the air b_3).

Using the relations $\sum_{i=1}^{3} b_i = 1$ and $u_2 \approx u$, and performing simple algebraic manipulations, we obtain the following system of differential equations:

$$\frac{\partial T}{\partial \tau} = K_{11}\nabla^2 T + K_{12}\nabla^2 u + K_{13}\nabla^2 P; \tag{19}$$

$$\frac{\partial u}{\partial \tau} = K_{21}\nabla^2 T + K_{22}\nabla^2 u + K_{23}\nabla^2 P; \tag{20}$$

$$\frac{\partial P}{\partial \tau} = K_{31}\nabla^2 T + K_{32}\nabla^2 u + K_{33}\nabla^2 P, \tag{21}$$

in which the coefficients K_{ij} (i, j = 1, 2, 3) are equal to

$$K_{11} = a + \frac{\varepsilon r a_m \delta}{c}, \quad K_{12} = \frac{\varepsilon r}{c} \, a_m, \quad K_{13} = \frac{\varepsilon r}{c} \, a_m \delta_p; \tag{22}$$

$$K_{21} = a_m \delta, \quad K_{22} = a_m, \quad K_{23} = a_m \delta_p; \tag{23}$$

$$K_{31} = a_m \delta \left(\frac{P \varepsilon r}{cT} + \beta - \frac{\varepsilon}{C_{\mathrm{vm}}} \right); \; K_{32} = a_m \left(\frac{\varepsilon r P}{cT} + \beta - \frac{\varepsilon}{C_{\mathrm{vm}}} \right); \tag{24}$$

$$K_{33} = a_p + a_m \delta_p \left(\frac{\varepsilon r P}{cT} + \beta - \frac{\varepsilon}{C_{\mathrm{vm}}} \right), \tag{25}$$

where C_{vm} is the specific capacity of the vaporous moisture (capacity of the capillary body with respect to the moist air):

$$C_{\mathrm{vm}} = \frac{M\Pi b}{\rho_0 RT}; \tag{26}$$

a_p is the coefficient of convective diffusion ($a_p = k_p/C_{\mathrm{vm}}\rho_0$); β is a coefficient, which depends on the porosity Π and the moisture content u:

$$\beta = \frac{P\rho_0}{\rho_2 \Pi - \rho_0 u}, \tag{27}$$

it takes into account how the fraction of the pores and capillaries of the body filled with moist air varies as a function of the moisture content of the body; δ_p is the relative coefficient of filtrational flow of vaporous moisture:

$$\delta_p = k_p / a_m \rho_0. \tag{28}$$

In deriving the heat transfer equation (19) we made the assumption that the convective heat transfer in the pores and capillaries is a small quantity, such that it could be neglected.

If we take T, θ, and P as the moisture transfer potentials, the system of differential equations of heat and mass transfer assumes the form

$$\frac{\partial T}{\partial \tau} = K'_{11}\nabla^2 T + K'_{12}\nabla^2\theta + K'_{13}\nabla^2 P; \tag{29}$$

$$\frac{\partial \theta}{\partial \tau} = K'_{21}\nabla^2 T + K'_{22}\nabla^2\theta + K'_{23}\nabla^2 P; \tag{30}$$

$$\frac{\partial P}{\partial \tau} = K'_{31}\nabla^2 T + K'_{32}\nabla^2\theta + K'_{33}\nabla^2 P; \tag{31}$$

where the coefficients K'_{ij} are given by

$$K'_{11} = a' + \frac{\varepsilon r C_m a'_m}{c'}\delta' = a + \frac{\varepsilon r C_m a_m}{c}\delta_\theta; \tag{32}$$

$$K'_{12} = \frac{\varepsilon r C_m a'_m}{c'} = \frac{\varepsilon r C_m a_m}{c}; \quad K'_{13} = \frac{\varepsilon r C_m}{c'} a'_m \delta'_p = \frac{\varepsilon r C_m}{c} a_m \delta'_p;$$

$$K'_{21} = a'_m \delta_\theta + a\theta'_T; \quad K'_{22} = a'_m, \quad K'_{23} = a_m \delta'_p; \tag{33}$$

$$K'_{31} = a_m \delta_\theta C_m \left(\beta - \frac{\varepsilon}{C_{\mathrm{vm}}} + \frac{\varepsilon r P}{cT}\right) + \frac{aP}{T}, \quad K'_{32} = a_m C_m \left(\beta - \frac{\varepsilon}{C_{\mathrm{vm}}} + \frac{\varepsilon r P}{cT}\right);$$

$$K'_{33} = a_p + a_m C_m \delta'_p \left(\beta - \frac{\varepsilon}{C_{\mathrm{vm}}} + \frac{\varepsilon r P}{cT}\right), \tag{34}$$

the relative coefficient δ'_p of the filtrational flow of moisture is defined in terms of the moisture conductivity coefficient λ_m:

$$\delta'_p = \frac{k_p}{\lambda_m} = \frac{\delta_p}{C_m}. \tag{35}$$

Comparing the coefficient K_{ij} and K'_{ij}, we note that in the coefficients K'_{ij} there appears, as an additional factor, the specific isothermal mass capacity C_m as the transfer coefficient from the moisture content u to the moisture transfer potential. Moreover, it is important to note that in deriving the system of equations (29)-(31), we did not assume that the temperature coefficient of the moisture transfer potential was equal to zero ($\theta'_T \neq 0$). It is then completely natural that the thermal gradient coefficients, $\delta(\delta \equiv \delta_u)$ and δ_θ, have different values, since they are referred to different vapor transfer potentials for the same temperature drop. In contrast to the coefficients δ_p and δ'_p, the coefficients δ_θ and δ are not directly proportional to each other (see relation (13)).

Once again, it should be remarked that at times the primes on the coefficients K_{ij} in the system of equations (29)-(31) will be omitted; however, this does not mean that they are equal to the coefficients K_{ij} in the system of equations (19)-(21). It can also be remarked that Yu. A. Mikhailov's calculations [3] show that the coefficients β and $\varepsilon rP/cT$ are significantly less than the coefficient $\varepsilon/C_{\mathrm{vm}}$ for a large number of moist materials; therefore, we can sometimes simplify the expressions for the coefficients K_{3j}; however, the most reliable way is to determine all of the transfer coefficients experimentally.

3. In making approximate calculations, we can, in practice, simplify the transfer equations (21) and (31). Since the mass content of vapor and air in the capillaries and pores of the body is negligibly small in comparison with the mass content of liquid ($u_1 + u_3 \ll u_2$), we can set the left side of equation (17) equal to zero. From equation (17) we then obtain

$$\varepsilon\rho_0 \frac{\partial u}{\partial \tau} = k_p \nabla^2 P. \tag{36}$$

Physically, this means that the total pressure drop inside the body arises only at the expense of evaporation of the liquid and of the presence inside the body of a resistance to the vapor motion (resistance to the filtrational flow of moisture).

The relation (36) allows us to eliminate the terms K_{i3} in the system of equations (19)-(21) and to reduce it to the system (1)-(2), and to reduce the system of equations (29)-(31) to the system (7)-(8). Therefore, the system of equations most widely used in drying technology is the system of equations (1)-(2) or the system (7)-(8), analogous to it.

These simplifications can be extended into regions of the body in a moist condition. For a moisture content of the body larger than hydroscopic, the vapor pressure of the material does not depend on the moisture content but only on the temperature (the pressure of the saturated vapor is a single-valued function of the temperature); then for a sufficiently intense evaporation of liquid inside the body, the total pressure P inside the body is a function of the temperature only (P = f(T)). As an approximation, we can write

$$\varepsilon\rho_0 \frac{\partial u}{\partial \tau} = k_p \left(\frac{\partial P}{\partial T}\right) \nabla^2 T. \tag{37}$$

In this case the heat transfer differential equation becomes the usual Fourier differential equation of heat conduction with an effective thermal conductivity coefficient taking account of the expenditure of heat in the evaporation of liquid inside the body (for details, see [4]). However, for an intense evaporation of liquid inside the body and large moisture flows, it is necessary to take into account the convective component of heat transfer inside the body. In this case the equation of heat transfer has the form

$$c\rho_0 \frac{\partial T}{\partial \tau} = \lambda \nabla^2 T + \varepsilon r \rho_0 \frac{\partial u}{\partial \tau} + \sum_i c_{pi}\, \mathbf{j}_{mi} \nabla T. \tag{38}$$

Moreover, from all the mass fluxes $\mathbf{j}_{mi}$ (i = 1, 2, 3) a principal one is selected, depending on the regimes of the drying parameters.

4. In conclusion, it should be remarked that a criterion ε for a phase transformation is the nonstationary moisture transfer characteristic. It was introduced as the characteristic of a vaporous moisture source according to the relation

$$I_{12} = -I_{21} = \varepsilon\rho_0 \frac{\partial u}{\partial \tau}. \tag{39}$$

For stationary moisture transfer the expression for the source must be different, since in this case $\partial u/\partial \tau \to 0$, $\varepsilon \to \infty$, and, consequently, the value of the source in relation (39) has an indeterminacy. This indeterminacy is easily resolved; we then obtain

$$I_{12} = -I_{21} = \operatorname{div} \mathbf{j}_1 = a_{m1}\rho_0 \nabla^2 u + a_{m1}\rho_0 \delta_1 \nabla^2 T. \tag{40}$$

The quantity $a_{m1}\delta_1$ is the thermal diffusion coefficient for the diffusion of the vaporous moisture into the capillary porous body ($a_{m1}^T = a_{m1}\delta_1$). It is assumed here that the coefficients a_{m1}, δ_1 do not depend on the coordinates. The relation (40) is the most general expression for the vaporous moisture source I_{12}; it is even valid for nonstationary moisture transfer. We obtain the expression (39) from it as a special case. Then, in place of the system of equations (1)-(2), we shall have a system of differential equations analogous to the system of equations (19)-(20) without the terms $\nabla^2 P$:

$$\frac{\partial T}{\partial \tau} = K_{11}'' \nabla^2 T + K_{12}'' \nabla^2 u; \tag{41}$$

$$\frac{\partial u}{\partial \tau} = K_{21}'' \nabla^2 T + K_{22}'' \nabla^2 u, \tag{42}$$

where the coefficients K_{ij}'' (i, j = 1, 2) are equal to

$$K_{11}'' = a + a_{m1}\delta_1 \frac{r_{12}}{c} = a + a_{m1}^T \frac{r_{12}}{c}, \quad K_{12}'' = a_{m1} \frac{r_{12}}{c}; \tag{43}$$

$$K_{21}'' = a_m \delta, \quad K_{22}'' = a_m. \tag{44}$$

Comparing the relations (43)-(44) with the relations (22)-(23), we find that $K''_{21} = K_{21}$, $K''_{22} = K_{22}$; the coefficients K''_{11} and K''_{12} are, respectively, equal to the coefficients K_{11} and K_{12} if we put $\varepsilon = a_{m1}/a_m$ and $\delta = \delta_1$ (for details concerning such assumptions, see [6]).

Thus the system of differential equations (41)-(42) of heat and mass transfer can be used for calculations of moisture and heat transfer in capillary porous bodies for arbitrary changes in u and T, including, in fact, stationary processes. The solution of the system of equations (41)-(42) is used in structural thermophysics, in the calculations of a number of chemical—industrial processes, and also in the treatment of experimental methods of determining the thermophysical properties of moist materials. Quite naturally, the system of equations (41)-(42) can be generalized by introducing the additional moisture transfer potential P. As a result, we obtain the system of equations (19)-(21) or the system (29)-(31) in which, in the formulas for the coefficients K_{ij} and K'_{ij}, there appear, in place of the criterion ε, the coefficients of diffusion a_{m1} and thermal diffusion ($a^T_{m1} = a_{m1}\delta_1$) of the vaporous moisture.

Thus the systems of differential equations of heat and mass transfer, (19)-(21) or (29)-(31), are the most general systems of equations for the diffusion transfer of heat and moisture in capillary porous bodies with kinetic coefficients in which, instead of the phase transformation coefficient, there appear coefficients of diffusion and thermal diffusion or the coefficient for the filtrational transfer of vaporous moisture.

NOTATION

u	is the moisture content of body;
T	is the temperature;
P	is the total pressure of air inside body;
c	is the specific heat capacity of moist body;
c_0	is the specific heat capacity of absolute dry body;
c_l	is the specific heat capacity of fluid;
$a = \lambda/c\rho_0$	is the thermal diffusivity of moist body;
λ	is the thermal conductivity of body;
ρ_0	is the density of absolute dry body;
τ	is the time;
θ	is the mass (moisture) transfer potential;
C_m	is the specific isothermal mass capacity of body;
θ'_T	is the temperature coefficient of mass transfer;
r	is the specific heat of evaporation ($r \equiv r_{12} \equiv r_{21}$);
ε	is the criterion of phase transition of fluid into vapor;
δ	is the thermo-gradient coefficient referred to moisture content difference;
λ_m	is the moisture conductivity;
M	is the molecular mass of humid air;
Π	is the porosity;
b	is the saturation of pores and capillaries of body with moisture;
R	is the universal gas constant;
c_p	is the isobaric heat capacity;
C_{vm}	is the specific capacity of vaporous moisture;
a_p	is the convective diffusivity;
k_p	is the coefficient of filtrational transfer of vaporous moisture;
δ_p	is the relative coefficient of filtrational flow of vaporous moisture;
ρ	is the density or concentration;
β	is the coefficient determined by formula (27) Remaining designations are given in text.

Subscripts

0	denotes the dry body skeleton;
1	denotes the vapor;
2	denotes the liquid;
3	denotes the air;
m	denotes the moisture mass flow (i, j = 1, 2, 3);
12	denotes the transition from vapor into liquid (condensation);
21	denotes the liquid evaporation (in formulas (39), (40), and (43)).

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